Solution Suppose he makes x suits, y jackets, and z pairs of slacks. Then his profit will be

$$P = 20x + 14y + 12z.$$

The constraints posed in the problem are

$$\begin{aligned} x &\geq 0, & x &\leq 20, \\ y &\geq 0, & y &\leq 30, \\ z &\geq 0, & z &\leq 40, \end{aligned}$$

$$6x + 3y + 2z \leq 230.$$

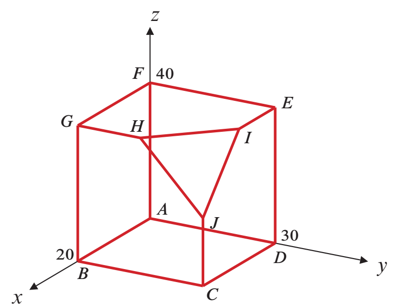

Figure 13.12 The convex set of points satisfying the constraints in Example 5

The last inequality is due to the limited supply of fabric. The solution set is shown in Figure 13.12. It has 10 vertices, $A, B, \ldots, J$. Since P increases in the direction of the vector $\nabla P = 20\mathbf{i} + 14\mathbf{j} + 12\mathbf{k}$, which points into the first octant, its maximum value cannot occur at any of the vertices $A, B, \ldots, G$. (Think about why.) Thus, we need look only at the vertices H, I, and J.

$$\begin{aligned} H &= (20, 10, 40), & P &= 1{,}020 \text{ at } H. \\ I &= (10, 30, 40), & P &= 1{,}100 \text{ at } I. \\ J &= (20, 30, 10), & P &= 940 \text{ at } J. \end{aligned}$$

Thus, the tailor should make 10 suits, 30 jackets, and 40 pairs of slacks to realize the maximum profit, \$1,100, from the fabric.

EXERCISES 13.2

1. Find the maximum and minimum values of $f(x, y) = x - x^2 + y^2$ on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

2. Find the maximum and minimum values of $f(x, y) = xy - 2x$ on the rectangle $-1 \leq x \leq 1, 0 \leq y \leq 1$.

3. Find the maximum and minimum values of $f(x, y) = xy - y^2$ on the disk $x^2 + y^2 \leq 1$.

4. Find the maximum and minimum values of $f(x, y) = x + 2y$ on the disk $x^2 + y^2 \leq 1$.

5. Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ over the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

6. Find the maximum and minimum values of $f(x, y) = xy(1 - x - y)$ over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

7. Find the maximum and minimum values of $f(x, y) = \sin x \cos y$ on the closed triangular region bounded by the coordinate axes and the line $x + y = 2\pi$.

8. Find the maximum value of $f(x, y) = \sin x \sin y \sin(x + y)$ over the triangle bounded by the coordinate axes and the line $x + y = \pi$.

9. The temperature at all points in the disk $x^2 + y^2 \leq 1$ is given by $T = (x + y)\, e^{-x^2 - y^2}$. Find the maximum and minimum temperatures at points of the disk.

10. Find the maximum and minimum values of

$$f(x, y) = \frac{x - y}{1 + x^2 + y^2}$$

on the upper half-plane $y \geq 0$.

11. Find the maximum and minimum values of $xy^2 + yz^2$ over the ball $x^2 + y^2 + z^2 \leq 1$.

12. Find the maximum and minimum values of $xz + yz$ over the ball $x^2 + y^2 + z^2 \leq 1$.

13. Consider the function $f(x, y) = xy\, e^{-xy}$ with domain the first quadrant: $x \geq 0$, $y \geq 0$. Show that $\lim_{x \to \infty} f(x, kx) = 0$. Does f have a limit as (x, y) recedes arbitrarily far from the origin in the first quadrant? Does f have a maximum value in the first quadrant?

14. Repeat Exercise 13 for the function $f(x, y) = xy^2\, e^{-xy}$.

15. In a certain community there are two breweries in competition, so that sales of each negatively affect the profits of the other. If brewery A produces x litres of beer per month and brewery B produces y litres per month, then brewery A's monthly profit \$$P$ and brewery B's monthly profit \$$Q$ are assumed to be

$$P = 2x - \frac{2x^2 + y^2}{10^6},$$

$$Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}.$$

Find the sum of the profits of the two breweries if each brewery independently sets its own production level to maximize its own profit and assumes its competitor does likewise. Find the sum of the profits if the two breweries cooperate to determine their respective productions to maximize that sum.

16. Equal angle bends are made at equal distances from the two ends of a 100 m long straight length of fence so the resulting three-segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?

17. Maximize $Q(x, y) = 2x + 3y$ subject to the constraints $x \geq 0$, $y \geq 0$, $y \leq 5$, $x + 2y \leq 12$, and $4x + y \leq 12$.

18. Minimize $F(x, y, z) = 2x + 3y + 4z$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y \geq 2$, $y + z \geq 2$, and $x + z \geq 2$.

19. A textile manufacturer produces two grades of fabric containing wool, cotton, and polyester. The deluxe grade has composition (by weight) 20% wool, 50% cotton, and 30% polyester, and it sells for \$3 per kilogram. The standard grade has composition 10% wool, 40% cotton, and 50% polyester, and it sells for \$2 per kilogram. If he has in stock 2,000 kg of wool and 6,000 kg each of cotton and polyester, how many kilograms of fabric of each grade should he manufacture to maximize his revenue?

20. A 10-hectare parcel of land is zoned for building densities of 6 detached houses per hectare, 8 duplex units per hectare, or 12 apartments per hectare. The developer who owns the land can make a profit of \$40,000 per house, \$20,000 per duplex unit, and \$16,000 per apartment that he builds. Municipal bylaws require him to build at least as many apartments as the total of houses and duplex units. How many of each type of dwelling should he build to maximize his profit?

13.3 Lagrange Multipliers

A constrained extreme-value problem is one in which the variables of the function to be maximized or minimized are not completely independent of one another, but must satisfy one or more constraint equations or inequalities. For instance, the problems

$$\textbf{maximize} \quad f(x, y) \quad \textbf{subject to} \quad g(x, y) = C$$

and

$$\begin{aligned} \textbf{minimize} \quad f(x, y, z, w) \quad \textbf{subject to} \quad & g(x, y, z, w) = C_1, \\ \textbf{and} \quad & h(x, y, z, w) = C_2 \end{aligned}$$

have, respectively, one and two constraint equations, while the problem

$$\textbf{maximize} \quad f(x, y, z) \quad \textbf{subject to} \quad g(x, y, z) \leq C$$

has a single constraint inequality.

Generally, inequality constraints can be regarded as restricting the domain of the function to be extremized to a smaller set that still has interior points. Section 13.2 was devoted to such problems. In each of the first three examples of that section we looked for *free* (i.e., *unconstrained*) extreme values in the interior of the domain, and we also examined the boundary of the domain, which was specified by one or more *constraint equations*. In Example 1 we parametrized the boundary and expressed the function to be extremized as a function of the parameter, thus reducing the boundary case to a free problem in one variable instead of a constrained problem in two variables. In Example 2 the boundary consisted of three line segments, on two of which the function was obviously zero. We solved the equation for the third boundary segment for y in terms of x, again in order to express the values of $f(x, y)$ on that segment as a function of one free variable. A similar approach was used in Example 3 to deal with the triangular boundary of the domain of the area function $A(\theta, \phi)$.

The reduction of extremization problems with equation constraints to free problems with fewer independent variables is only feasible when the constraint equations can be solved either explicitly for some variables in terms of others or parametrically for all variables in terms of some parameters. It is often very difficult or impossible to solve the constraint equations, so we need another technique.

The Method of Lagrange Multipliers

A technique for finding extreme values of $f(x, y)$ subject to the equality constraint $g(x, y) = 0$ is based on the following theorem:

Section 13.2 Extreme Values of Functions Defined on Restricted Domains (page 758)

1. $f(x, y) = x - x^2 + y^2$ on
$R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 1\}$.
For critical points:

$$0 = f_1(x, y) = 1 - 2x, \qquad 0 = f_2(x, y) = 2y.$$

The only CP is $(1/2, 0)$, which lies on the boundary of R.
The boundary consists of four segments; we investigate each.
On $x = 0$ we have $f(x, y) = f(0, y) = y^2$ for $0 \le y \le 1$, which has minimum value 0 and maximum value 1.
On $y = 0$ we have $f(x, y) = f(x, 0) = x - x^2 = g(x)$ for $0 \le x \le 2$. Since $g'(x) = 1 - 2x = 0$ at $x = 1/2$, $g(1/2) = 1/4$, $g(0) = 0$, and $g(2) = -2$, the maximum and minimum values of f on the boundary segment $y = 0$ are 1/4 and -2 respectively.
On $x = 2$ we have $f(x, y) = f(2, y) = -2 + y^2$ for $0 \le y \le 1$, which has minimum value -2 and maximum value -1.
On $y = 1$, $f(x, y) = f(x, 1) = x - x^2 + 1 = g(x) + 1$ for $0 \le x \le 2$. Thus the maximum and minimum values of f on the boundary segment $y = 1$ are 5/4 and -1 respectively.
Overall, f has maximum value $5/4$ and minimum value -2 on the rectangle R.

2. $f(x, y) = xy - 2x$ on
$R = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 1\}$.
For critical points:

$$0 = f_1(x, y) = y - 2, \qquad 0 = f_2(x, y) = x.$$

The only CP is $(0, 2)$, which lies outside R. Therefore the maximum and minimum values of f on R lie on one of the four boundary segments of R.
On $x = -1$ we have $f(-1, y) = 2 - y$ for $0 \le y \le 1$, which has maximum value 2 and minimum value 1.
On $x = 1$ we have $f(1, y) = y - 2$ for $0 \le y \le 1$, which has maximum value -1 and minimum value -2.
On $y = 0$ we have $f(x, 0) = -2x$ for $-1 \le x \le 1$, which has maximum value 2 and minimum value -2.
On $y = 1$ we have $f(x, 1) = -x$ for $-1 \le x \le 1$, which has maximum value 1 and minimum value -1.
Thus the maximum and minimum values of f on the rectangle R are 2 and -2 respectively.

3. $f(x, y) = xy - y^2$ on $D = \{(x, y) : x^2 + y^2 \le 1\}$.
For critical points:

$$0 = f_1(x, y) = y, \qquad 0 = f_2(x, y) = x - 2y.$$

The only CP is $(0,0)$, which lies inside D. We have $f(0,0)=0$.
The boundary of D is the circle $x=\cos t$, $y=\sin t$, $-\pi\le t\le\pi$. On this circle we have

$$\begin{aligned}g(t)&=f(\cos t,\sin t)=\cos t\sin t-\sin^2 t\\&=\frac{1}{2}\Big[\sin 2t+\cos 2t-1\Big],\qquad(-\pi\le t\le\pi).\\g(0)&=g(2\pi)=0\\g'(t)&=\cos 2t-\sin 2t.\end{aligned}$$

The critical points of g satisfy $\cos 2t=\sin 2t$, that is, $\tan 2t=1$, so $2t=\pm\frac{\pi}{4}$ or $\pm\frac{5\pi}{4}$, and $t=\pm\frac{\pi}{8}$ or $\pm\frac{5\pi}{8}$. We have

$$\begin{aligned}g\left(\frac{\pi}{8}\right)&=\frac{1}{2\sqrt{2}}-\frac{1}{2}+\frac{1}{2\sqrt{2}}=\frac{1}{\sqrt{2}}-\frac{1}{2}>0\\g\left(-\frac{\pi}{8}\right)&=-\frac{1}{2\sqrt{2}}-\frac{1}{2}+\frac{1}{2\sqrt{2}}=-\frac{1}{2}\\g\left(\frac{5\pi}{8}\right)&=-\frac{1}{2\sqrt{2}}-\frac{1}{2}-\frac{1}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}-\frac{1}{2}\\g\left(-\frac{5\pi}{8}\right)&=\frac{1}{2\sqrt{2}}-\frac{1}{2}-\frac{1}{2\sqrt{2}}=-\frac{1}{2}.\end{aligned}$$

Thus the maximum and minimum values of f on the disk D are $\frac{1}{\sqrt{2}}-\frac{1}{2}$ and $-\frac{1}{\sqrt{2}}-\frac{1}{2}$ respectively.

4. $f(x,y)=x+2y$ on the closed disk $x^2+y^2\le 1$. Since $f_1=1$ and $f_2=2$, f has no critical points, and the maximum and minimum values of f, which must exist because f is continuous on a closed, bounded set in the plane, must occur at boundary points of the domain, that is, points of the circle $x^2+y^2=1$. This circle can be parametrized $x=\cos t$, $y=\sin t$, so that

$$f(x,y)=f(\cos t,\sin t)=\cos t+2\sin t=g(t),\ \text{say.}$$

For critical points of g: $0=g'(t)=-\sin t+2\cos t$. Thus $\tan t=2$, and $x=\pm 1/\sqrt{5}$, $y=\pm 2/\sqrt{5}$. The critical points are $(-1/\sqrt{5},-2/\sqrt{5})$, where f has value $-\sqrt{5}$, and $(1/\sqrt{5},2/\sqrt{5})$, where f has value $\sqrt{5}$. Thus the maximum and minimum values of $f(x,y)$ on the disk are $\sqrt{5}$ and $-\sqrt{5}$ respectively.

5. $f(x,y)=xy-x^3y^2$ on the square S: $0\le x\le 1$, $0\le y\le 1$.
$f_1=y-3x^2y^2=y(1-3x^2y)$,
$f_2=x-2x^3y=x(1-2x^2y)$.
$(0,0)$ is a critical point. Any other critical points must satisfy $3x^2y=1$ and $2x^2y=1$, that is, $x^2y=0$. Therefore $(0,0)$ is the only critical point, and it is on the boundary of S. We need therefore only consider the values of f on the boundary of S.
On the sides $x=0$ and $y=0$ of S, $f(x,y)=0$.
On the side $x=1$ we have $f(1,y)=y-y^2=g(y)$, $(0\le y\le 1)$. g has maximum value $1/4$ at its critical point $y=1/2$.
On the side $y=1$ we have $f(x,1)=x-x^3=h(x)$, $(0\le x\le 1)$. h has critical point given by $1-3x^2=0$; only $x=1/\sqrt{3}$ is on the side of S.
$h\left(\frac{1}{\sqrt{3}}\right)=\frac{2}{3\sqrt{3}}>\frac{1}{4}$.
On the square S, $f(x,y)$ has minimum value 0 (on the sides $x=0$ and $y=0$ and at the corner $(1,1)$ of the square), and maximum value $2/(3\sqrt{3})$ at the point $(1/\sqrt{3},1)$. There is a smaller local maximum value at $(1,1/2)$.

6. $f(x,y)=xy(1-x-y)$ on the triangle T shown in the figure. Evidently $f(x,y)=0$ on all three boundary segements of T, and $f(x,y)>0$ inside T. Thus the minimum value of f on T is 0, and the maximum value must occur at an interior critical point. For critical points:

$$0=f_1(x,y)=y(1-2x-y),\qquad 0=f_2(x,y)=x(1-x-2y).$$

The only critical points are $(0,0)$, $(1,0)$ and $(0,1)$, which are on the boundary of T, and $(1/3,1/3)$, which is inside T. The maximum value of f over T is $f(1/3,1/3)=1/27$.

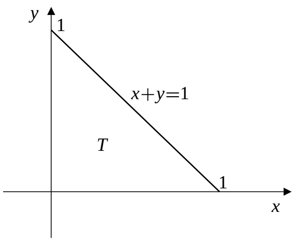

Fig. 13.2.6

7. Since $-1\le f(x,y)=\sin x\cos y\le 1$ everywhere, and since $f(\pi/2,0)=1$, $f(3\pi/2,0)=-1$, and both $(\pi/2,0)$ and $(3\pi/2,0)$ belong to the triangle bounded by $x=0$, $y=0$ and $x+y=2\pi$, therefore the maximum and minimum values of f over that triangle must be 1 and -1 respectively.

8. $f(x,y)=\sin x\sin y\sin(x+y)$ on the triangle T shown in the figure. Evidently $f(x,y)=0$ on the boundary of T, and $f(x,y)>0$ at all points inside T. Thus the minimum value of f on T is zero, and the maximum value must occur at an interior critical point. For critical points inside T we must have

$$\begin{aligned}0&=f_1(x,y)=\cos x\sin y\sin(x+y)+\sin x\sin y\cos(x+y)\\0&=f_2(x,y)=\sin x\cos y\sin(x+y)+\sin x\sin y\cos(x+y).\end{aligned}$$

Therefore $\cos x \sin y = \cos y \sin x$, which implies $x = y$ for points inside T, and

$$\begin{aligned}&\cos x \sin x \sin 2x + \sin^2 x \cos 2x = 0\\&2\sin^2 x\cos^2 x + 2\sin^2 x\cos^2 x - \sin^2 x = 0\\&4\cos^2 x = 1.\end{aligned}$$

Thus $\cos x = \pm 1/2$, and $x = \pm\pi/3$. The interior critical point is $(\pi/3, \pi/3)$, where f has the value $3\sqrt{3}/8$. This is the maximum value of f on T.

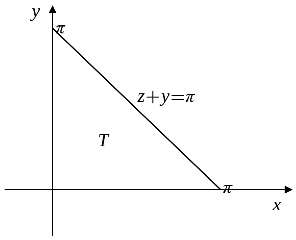

Fig. 13.2.8

9. $T = (x+y)e^{-x^2-y^2}$ on $D = \{(x, y) : x^2 + y^2 \le 1\}$.
For critical points:

$$\begin{aligned}0 &= \frac{\partial T}{\partial x} = \Big(1 - 2x(x+y)\Big)e^{-x^2-y^2}\\0 &= \frac{\partial T}{\partial y} = \Big(1 - 2y(x+y)\Big)e^{-x^2-y^2}.\end{aligned}$$

The critical points are given by
$2x(x+y) = 1 = 2y(x+y)$, which forces $x = y$ and $4x^2 = 1$, so $x = y = \pm\frac{1}{2}$.
The two critical points are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, both of which lie inside D. T takes the values $\pm e^{-1/2}$ at these points.

On the boundary of D, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, so that

$$T = (\cos t + \sin t)e^{-1} = g(t), \qquad (0 \le t \le 2\pi).$$

We have $g(0) = g(2\pi) = e^{-1}$. For critical points of g:

$$0 = g'(t) = (\cos t - \sin t)e^{-1},$$

so $\tan t = 1$ and $t = \pi/4$ or $t = 5\pi/4$. Observe that $g(\pi/4) = \sqrt{2}e^{-1}$, and $g(5\pi/4) = -\sqrt{2}e^{-1}$.
Since $e^{-1/2} > \sqrt{2}e^{-1}$ (because $e > 2$), the maximum and minimum values of T on the disk are $\pm e^{-1/2}$, the values at the interior critical points.

10. $f(x, y) = \dfrac{x-y}{1+x^2+y^2}$ on the half-plane $y \ge 0$.
For critical points:

$$\begin{aligned}0 &= f_1(x, y) = \frac{1 - x^2 + y^2 + 2xy}{(1+x^2+y^2)^2}\\0 &= f_2(x, y) = \frac{-1 - x^2 + y^2 - 2xy}{(1+x^2+y^2)^2}.\end{aligned}$$

Any critical points must satisfy $1 - x^2 + y^2 + 2xy = 0$ and $-1 - x^2 + y^2 - 2xy = 0$, and hence $x^2 = y^2$ and $2xy = -1$. Therefore $y = -x = \pm 1/\sqrt{2}$. The only critical point in the region $y \ge 0$ is $(-1/\sqrt{2}, 1/\sqrt{2})$, where f has the value $-1/\sqrt{2}$.

On the boundary $y = 0$ we have

$$f(x, 0) = \frac{x}{1+x^2} = g(x), \qquad (-\infty < x < \infty).$$

Evidently, $g(x) \to 0$ as $x \to \pm\infty$.
Since $g'(x) = \dfrac{1-x^2}{(1+x^2)^2}$, the critical points of g are $x = \pm 1$. We have $g(\pm 1) = \pm\frac{1}{2}$.
The maximum and minimum values of f on the upper half-plane $y \ge 0$ are $1/2$ and $-1/\sqrt{2}$ respectively.

11. Let $f(x, y, z) = xy^2 + yz^2$ on the ball B: $x^2+y^2+z^2 \le 1$.
First look for interior critical points:

$$0 = f_1 = y^2, \quad 0 = f_2 = 2xy + z^2, \quad 0 = f_3 = 2yz.$$

All points on the x-axis are CPs, and $f = 0$ at all such points.

Now consider the boundary sphere $z^2 = 1 - x^2 - y^2$. On it

$$f(x, y, z) = xy^2 + y(1 - x^2 - y^2) = xy^2 + y - x^2y - y^3 = g(x, y),$$

where g is defined for $x^2 + y^2 \le 1$. Look for interior CPs of g:

$$\begin{aligned}0 &= g_1 = y^2 - 2xy = y(y - 2x)\\0 &= g_2 = 2xy + 1 - x^2 - 3y^2.\end{aligned}$$

Case I: $y = 0$. Then $g = 0$ and $f = 0$.
Case II: $y = 2x$. Then $4x^2 + 1 - x^2 - 12x^2 = 0$, so $9x^2 = 1$ and $x = \pm 1/3$. This case produces critical points

$$\left(\frac{1}{3}, \frac{2}{3}, \pm\frac{2}{3}\right), \quad \text{where } f = \frac{4}{9}, \quad \text{and}$$
$$\left(-\frac{1}{3}, -\frac{2}{3}, \pm\frac{2}{3}\right), \quad \text{where } f = -\frac{4}{9}.$$

Now we must consider the boundary $x^2 + y^2 = 1$ of the domain of g. Here

$$g(x, y) = xy^2 = x(1 - x^2) = x - x^3 = h(x)$$

for $-1 \le x \le 1$. At the endpoints $x = \pm 1$, $h = 0$, so $g = 0$ and $f = 0$. For CPs of h:

$$0 = h'(x) = 1 - 3x^2,$$

so $x = \pm 1/\sqrt{3}$ and $y = \pm\sqrt{2/3}$. The value of h at such points is $\pm 2/(3\sqrt{3})$. However $2/(3\sqrt{3}) < 4/9$, so the maximum value of f is 4/9, and the minimum value is $-4/9$.

12. Let $f(x, y, z) = xz + yz$ on the ball $x^2 + y^2 + z^2 \le 1$. First look for interior critical points:

$$0 = f_1 = z, \quad 0 = f_2 = z, \quad 0 = f_3 = x + y.$$

All points on the line $z = 0$, $x + y = 0$ are CPs, and $f = 0$ at all such points.

Now consider the boundary sphere $x^2 + y^2 + z^2 = 1$. On it

$$f(x, y, z) = (x+y)z = \pm(x+y)\sqrt{1 - x^2 - y^2} = g(x, y),$$

where g has domain $x^2 + y^2 \le 1$. On the boundary of its domain, g is identically 0, although g takes both positive and negative values at some points inside its domain. Therefore, we need consider only critical points of g in $x^2 + y^2 < 1$. For such CPs:

$$\begin{aligned} 0 = g_1 &= \sqrt{1 - x^2 - y^2} + \frac{(x+y)(-2x)}{2\sqrt{1 - x^2 - y^2}} \\ &= \frac{1 - x^2 - y^2 - x^2 - xy}{\sqrt{1 - x^2 - y^2}} \\ 0 = g_2 &= \frac{1 - x^2 - y^2 - xy - y^2}{\sqrt{1 - x^2 - y^2}}. \end{aligned}$$

Therefore $2x^2 + y^2 + xy = 1 = x^2 + 2y^2 + xy$, from which $x^2 = y^2$.
Case I: $x = -y$. Then $g = 0$, so $f = 0$.
Case II: $x = y$. Then $2x^2 + x^2 + x^2 = 1$, so $x^2 = 1/4$ and $x = \pm 1/2$. g (which is really two functions depending on our choice of the "+" or "−" sign) has four CPs, two corresponding to $x = y = 1/2$ and two to $x = y = -1/2$. The values of g at these four points are $\pm 1/\sqrt{2}$.

Since we have considered all points where f can have extreme values, we conclude that the maximum value of f on the ball is $1/\sqrt{2}$ (which occurs at the boundary points $\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$) and minimum value $-1/\sqrt{2}$ (which occurs at the boundary points $\pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$).

13. $f(x, y) = xye^{-xy}$ on $Q = \{(x, y) : x \ge 0,\ y \ge 0\}$.
Since $f(x, kx) = kx^2 e^{-kx^2} \to 0$ as $x \to \infty$ if $k > 0$, and $f(x, 0) = f(0, y) = 0$, we have $f(x, y) \to 0$ as (x, y) recedes to infinity along any straight line from the origin lying in the first quadrant Q.
However, $f\left(x, \dfrac{1}{x}\right) = 1$ and $f(x, 0) = 0$ for all $x > 0$, even though the points $\left(x, \dfrac{1}{x}\right)$ and $(x, 0)$ become arbitrarily close together as x increases. Thus f does not have a limit as $x^2 + y^2 \to \infty$.
Observe that $f(x, y) = re^{-r} = g(r)$ on the hyperbola $xy = r > 0$. Since $g(r) \to 0$ as r approaches 0 or ∞, and

$$g'(r) = (1 - r)e^{-r} = 0 \quad \Rightarrow \quad r = 1,$$

$f(x, y)$ is everywhere on Q less than $g(1) = 1/e$. Thus f does have a maximum value on Q.

14. $f(x, y) = xy^2 e^{-xy}$ on $Q = \{(x, y) : x \ge 0,\ y \ge 0\}$.
As in Exercise 13, $f(x, 0) = f(0, y) = 0$ and $\lim_{x\to\infty} f(x, kx) = k^2 x^3 e^{-x^2} = 0$.
Also, $f(0, y) = 0$ while $f\left(\dfrac{1}{y}, y\right) = \dfrac{y}{e} \to \infty$ as $y \to \infty$, so that f has no limit as $x^2 + y^2 \to \infty$ in Q, and f has no maximum value on Q.

15. If brewery A produces x litres per month and brewery B produces y litres per month, then the monthly profits of the two breweries are given by

$$P = 2x - \frac{2x^2 + y^2}{10^6}, \qquad Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}.$$

STRATEGY I. Each brewery selects its production level to maximize its own profit, and assumes its competitor does the same.
Then A chooses x to satisfy

$$0 = \frac{\partial P}{\partial x} = 2 - \frac{4x}{10^6} \quad \Rightarrow \quad x = 5 \times 10^5.$$

B chooses y to satisfy

$$0 = \frac{\partial Q}{\partial y} = 2 - \frac{8y}{2 \times 10^6} \quad \Rightarrow \quad y = 5 \times 10^5.$$

The total profit of the two breweries under this strategy is

$$\begin{aligned} P + Q &= 10^6 - \frac{3 \times 25 \times 10^{10}}{10^6} + 10^6 - \frac{5 \times 25 \times 10^{10}}{2 \times 10^6} \\ &= \$625,000. \end{aligned}$$

STRATEGY II. The two breweries cooperate to maximize the total profit

$$T = P + Q = 2x + 2y - \frac{5x^2 + 6y^2}{2 \times 10^6}$$

by choosing x and y to satisfy

$$\begin{aligned} 0 &= \frac{\partial T}{\partial x} = 2 - \frac{10x}{2\times 10^6}, \\ 0 &= \frac{\partial T}{\partial y} = 2 - \frac{12y}{2\times 10^6}. \end{aligned}$$

Thus $x = 4\times 10^5$ and $y = \dfrac{1}{3}\times 10^6$.
In this case the total monthly profit is

$$\begin{aligned} P + Q &= 8\times 10^5 + \frac{2}{3}\times 10^6 - \frac{80\times 10^{10} + \frac{2}{3}\times 10^{12}}{2\times 10^6} \\ &\approx \$733,333. \end{aligned}$$

Observe that the total profit is larger if the two breweries cooperate and fix prices to maximize it.

16. Let the dimensions be as shown in the figure. Then $2x + y = 100$, the length of the fence. For maximum area A of the enclosure we will have $x > 0$ and $0 < \theta < \pi/2$. Since $h = x\cos\theta$, the area A is

$$\begin{aligned} A &= xy\cos\theta + 2\times\frac{1}{2}(x\sin\theta)(x\cos\theta) \\ &= x(100-2x)\cos\theta + x^2\sin\theta\cos\theta \\ &= (100x - 2x^2)\cos\theta + \frac{1}{2}x^2\sin 2\theta. \end{aligned}$$

We look for a critical point of A satisfying $x > 0$ and $0 < \theta < \pi/2$.

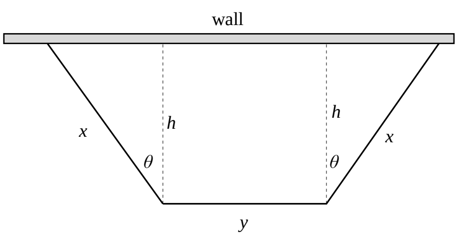

Fig. 13.2.16

$$\begin{aligned} 0 = \frac{\partial A}{\partial x} &= (100-4x)\cos\theta + x\sin 2\theta \\ &\Rightarrow \cos\theta(100 - 4x + 2x\sin\theta) = 0 \\ &\Rightarrow 4x - 2x\sin\theta = 100 \Rightarrow x = \frac{50}{2-\sin\theta} \\ 0 = \frac{\partial A}{\partial\theta} &= -(100x - 2x^2)\sin\theta + x^2\cos 2\theta \\ &\Rightarrow x(1 - 2\sin^2\theta) + 2x\sin\theta - 100\sin\theta = 0. \end{aligned}$$

Substituting the first equation into the second we obtain

$$\begin{aligned} &\frac{50}{2-\sin\theta}\Big(1 - 2\sin^2\theta + 2\sin\theta\Big) - 100\sin\theta = 0 \\ &50(1 - 2\sin^2\theta + 2\sin\theta) = 100(2\sin\theta - \sin^2\theta) \\ &50 = 100\sin\theta. \end{aligned}$$

Thus $\sin\theta = 1/2$, and $\theta = \pi/6$.
Therefore $x = \dfrac{50}{2-(1/2)} = \dfrac{100}{3}$, and
$y = 100 - 2x = \dfrac{100}{3}$.

The maximum area for the enclosure is

$$A = \left(\frac{100}{3}\right)^2\frac{\sqrt{3}}{2} + \left(\frac{100}{3}\right)^2\frac{1}{2}\frac{\sqrt{3}}{2} = \frac{2500}{\sqrt{3}}$$

square units. All three segments of the fence will be the same length, and the bend angles will be 120°.

17. To maximize $Q(x,y) = 2x + 3y$ subject to

$$x \ge 0, \quad y \ge 0, \quad y \le 5, \quad x + 2y \le 12, \quad 4x + y \le 12.$$

The constraint region is shown in the figure.

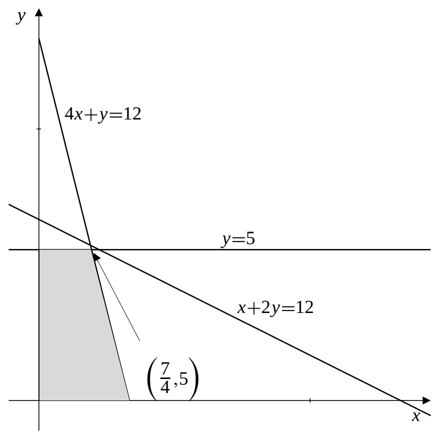

Fig. 13.2.17

Observe that any point satisfying $y \le 5$ and $4x + y \le 12$ automatically satisfies $x + 2y \le 12$. Since $y = 5$ and $4x + y = 12$ intersect at $\left(\dfrac{7}{4}, 5\right)$, the maximum value of $Q(x,y)$ subject to the given constraints is

$$Q\left(\frac{7}{4}, 5\right) = \frac{7}{2} + 15 = \frac{37}{2}.$$

18. Minimize $F(x,y,z) = 2x + 3y + 4z$ subject to

$$\begin{aligned} x &\ge 0, & y &\ge 0, & z &\ge 0, \\ x + y &\ge 2, & y + z &\ge 2, & x + z &\ge 2. \end{aligned}$$

Here the constraint region has vertices $(1, 1, 1)$, $(2, 2, 0)$, $(2, 0, 2)$, and $(0, 2, 2)$. Since $F(1, 1, 1) = 9$, $F(2, 2, 0) = 10$, $F(2, 0, 2) = 12$, and $F(0, 2, 2) = 14$, the minimum value of F subject to the constraints is 9.

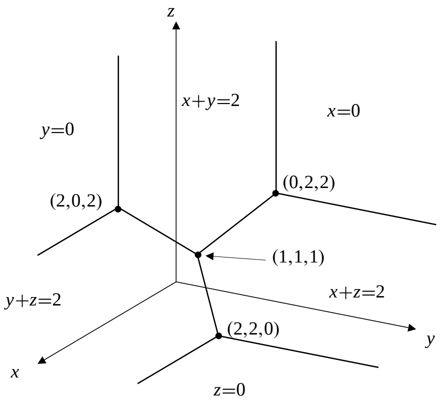

Fig. 13.2.18

19. Suppose that x kg of deluxe fabric and y kg of standard fabric are produced. Then the total revenue is

$$R = 3x + 2y.$$

The constraints imposed by raw material availability are

$$\begin{aligned} \frac{20}{100}x + \frac{10}{100}y \le 2,000, &\iff 2x + y \le 20,000 \\ \frac{50}{100}x + \frac{40}{100}y \le 6,000, &\iff 5x + 4y \le 60,000 \\ \frac{30}{100}x + \frac{50}{100}y \le 6,000, &\iff 3x + 5y \le 60,000. \end{aligned}$$

The lines $2x + y = 20,000$ and $5x + 4y = 60,000$ intersect at the point $\left(\frac{20,000}{3}, \frac{20,000}{3}\right)$, which satisfies $3x + 5y \le 60,000$, so lies in the constraint region. We have

$$f\left(\frac{20,000}{3}, \frac{20,000}{3}\right) \approx 33,333.$$

The lines $2x + y = 20,000$ and $3x + 5y = 60,000$ intersect at the point $\left(\frac{40,000}{7}, \frac{60,000}{7}\right)$, which does not satisfy $5x + 4y \le 60,000$ and so does not lie in the constraint region.

The lines $5x + 4y = 60,000$ and $3x + 5y = 60,000$ intersect at the point $\left(\frac{60,000}{13}, \frac{120,000}{13}\right)$, which satisfies $2x + y \le 20,000$ and so lies in the constraint region. We have

$$f\left(\frac{60,000}{13}, \frac{120,000}{13}\right) \approx 32,307.$$

To produce the maximum revenue, the manufacturer should produce $20,000/3 \approx 6,667$ kg of each grade of fabric.

20. If the developer builds x houses, y duplex units, and z apartments, his profit will be

$$P = 40,000x + 20,000y + 16,000z.$$

The legal constraints imposed require that

$$\frac{x}{6} + \frac{y}{8} + \frac{z}{12} \le 10, \quad \text{that is } 4x + 3y + 2z \le 240,$$

and also

$$z \ge x + y.$$

Evidently we must also have $x \ge 0$, $y \ge 0$, and $z \ge 0$. The planes $4x + 3y + 2z = 240$ and $z = x + y$ intersect where $6x + 5y = 240$. Thus the constraint region has vertices $(0, 0, 0)$, $(40, 0, 40)$, $(0, 48, 48)$, and $(0, 0, 120)$, which yield revenues of \$0, \$2,240,000, \$1,728,000, and \$1,920,000 respectively.

For maximum profit, the developer should build 40 houses, no duplex units, and 40 apartments.